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ON THE SHOCK TRANSITION, THE HYDRAULIC JUMP, AND VORTEX BREAKDOWN

by

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Abstract

The normal shock, hydraulic jump, and vortex breakdown have a common feature: they are all marked by a transition from a supercritical to a subcritical flow state. These phenomena are due to the nonlinearity of the flow, and it will be shown that viscosity also plays an essential role. This paper demonstrates explicitly how viscosity enters into these flows. The treatment of normal shock served a starting illustration. Then an equation governing the phenomenon of hydraulic jump is derived with the inclusion of the effect of viscosity. It is explicitly shown that supercritical flow is not stable and has to go through a transition to a conjugate subcritical flow state downstream. Similar treatment is also applied to vortex breakdown with largely similar results.

I. Introduction

The phenomena of normal shock, hydraulic jump and vortex breakdown have these same features: they are all marked by a transition from a supercritical to a subcritical flow state, and the mechanism of this transition is due to the nonlinearity of the flow and viscosity. In this paper, we wish to demonstrate explicitly how viscosity plays such a role. Our treatment of a normal shock is not new. It has been used for the study of shock structure(1). However, it has not heretofore been related as transparently to the intrinsic stability of subsonic flow and the intrinsic instability of the supersonic flow. The problem of the hydraulic jump is somewhat more involved. Not only is a sharp jump possible, but one may have oscillations. The inviscid theory has been fully explored by Serre(2), and Benjamin and Lighthill(3). However, to our knowledge this is the first time that the explicit transition is demonstrated. For the complex phenomenon of the vortex breakdown, we shall only study one phase of the problem, i.e., the breakdown and oscillation of its gross flows. We concur with the suggestions of Harvey(4) and Benjamin(5), that this phenomenon is somewhat similar to the hydraulic jump. The governing equation and the explicit demonstration of its transition are to our knowledge obtained for the first time here.

II. Shock in One Dimensional Flow of a Barotropic Fluid

Consider a barotropic fluid so that the pressure p is a function of density ρ only. For one-dimensional steady flow, the continuity equation becomes

$$\frac{d}{dx}(\rho u) = 0 \quad , \tag{1}$$

and the momentum equation becomes

$$u \frac{du}{dx} = - \frac{1}{\rho} \frac{dp}{dx} + \nu \frac{d^2u}{dx^2} \quad , \tag{2}$$

where u is the velocity, and ν the kinematic viscosity of the fluid.

Equation (1) leads to

$$\rho u = A \quad ,$$

where A is the constant flux of fluid across any section. Denote

$$h(\rho) = \int \frac{dp}{\rho} \quad , \tag{3}$$

then we obtain from Eq. (2):

$$\nu \frac{du}{dx} = f(u) = \frac{u^2}{2} + h\left(\frac{A}{u}\right) - B \quad , \tag{4}$$

where B is an integration constant, which may be related to values of u and $\frac{du}{dx}$ at some initial section.

The behavior of $f(u)$ depends on the barotropic relation $p = p(\rho)$. Now it is easy to see that

$$\frac{df}{du} = u - \frac{c^2}{u} \quad ,$$

where the sound speed c is given by $c^2 = \frac{dp}{d\rho}$. Moreover $u = c$ is a minimum so long as $1 + \frac{\rho}{c}\frac{dc}{d\rho}$ is positive, which is generally valid for ordinary fluids.

To be more specific, let us consider the barotropic relation

$$p = \alpha + \beta \rho^{\gamma} \quad ,$$

with $\gamma \geqslant 1$. Then the general behavior of $f(u)$ is shown in Fig. 1. We may note that

$$f(u) \sim u^2 , \quad \text{as } u \to \infty ,$$

and

$$f(u) \sim \begin{cases} u^{1-\gamma} & \text{for } \gamma > 1 \\ -\ln u & \text{for } \gamma = 1 \end{cases} \qquad \text{as } u \to 0 .$$

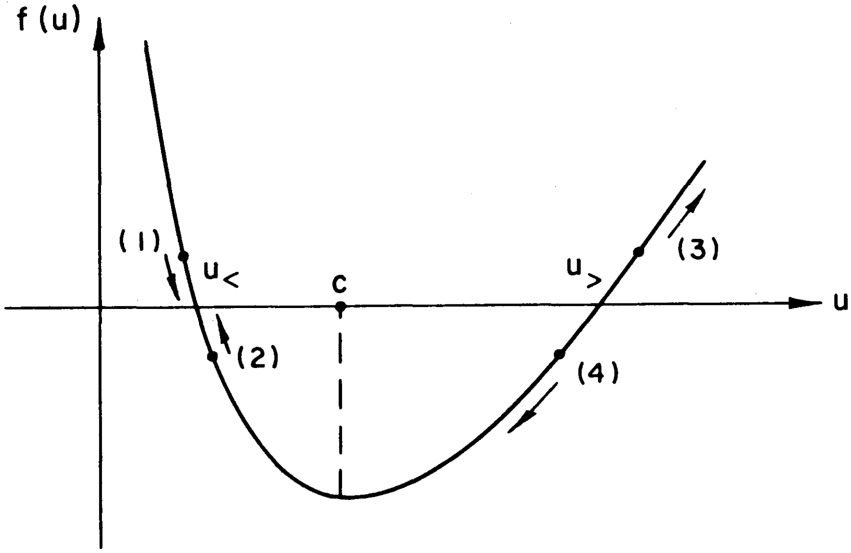

FIG. 1

Thus, $f(u)$ will have a negative portion, so long as the value of $\frac{du}{dx}$ at some initial section is not too large.

Of particular interest is the case where $\left(\frac{du}{dx}\right)_i$ is very small at the initial section. Then the initial velocity u_i will be close to either $u_<$ or $u_>$ according to Fig. 1, with $u_< < c$, and $u_> > c$.

With the aid of Fig. 1, the nature of the solution is clearly revealed If u_i is close to $u_<$, then $u(x)$ will be bounded in the neighborhood of $u_<$ for all subsequent x, since if $\left(\frac{du}{dx}\right)_i > 0$, the solution will follow arrow (1); and if $\left(\frac{du}{dx}\right)_i < 0$, the solution will follow arrow (2), all terminating asymptotically to $u_<$. On the other hand, if u_i is close to $u_>$, then $u(x)$ will never be bounded in the neighborhood of $u_>$. If $\left(\frac{du}{dx}\right)_i > 0$, then $u(x)$ following arrow (3) will increase indefinitely as x increases; and if $\left(\frac{du}{dx}\right)_i < 0$, $u(x)$, following arrow (4) in Fig. 1, will gradually decrease and terminate asymptotically to $u_<$. The former case represents an expansion flow, and the latter, a shock.

It may be noted that the inviscid theory will admit two possible solutions for our problem, i.e., $u = u_<$ and $u = u_>$. The existence and nature of the shock which bridge up two distinct uniform flow states are usually established by the consideration of entropy. The previous analysis, though it yields no new results, reveals explicitly the role played by the viscosity. Thus, while the subsonic uniform flow is always stable in the sense that the flow will always remain bounded in its neighborhood against possible external pertubations, the supersonic uniform flow is not stable. For supersonic initial flows, only two alternatives are permissible: the expansion flow, or the shock. It also automatically comes out that the latter will be a compression shock through which supersonic flow passes to subsonic flow.

We may also remark that the previous results depend on the generally valid condition that $1 + \frac{\rho}{c}\frac{dc}{d\rho}$ is positive. If $1 + \frac{\rho}{c}\frac{dc}{d\rho}$ were negative, then the supersonic uniform flow would be stable, and we could only have the expansion shock through which subsonic flow passes to supersonic flow.

III. Hydraulic Jump

Similar to the case of the one-dimensional shock, viscosity is also responsible for the stability of subcritical states and the transition, i.e. hydraulic jump, from supercritical to subcritical status for channel flows. The problem, however, is more involved in this case.

Consider the steady two-dimensional flow of an incompressible fluid under gravity. Let the gravitational force be pointing in -y direction, and let u and v be the velocity components in x and y directions respectively. Then we have the following governing equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad , \tag{5}$$

$$u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad , \tag{6}$$

and

$$u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - g + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \quad . \tag{7}$$

Equation (5) leads to the existence of a stream function ψ, such that

$$u = \frac{\partial \psi}{\partial y} \quad ; \quad v = -\frac{\partial \psi}{\partial x} \quad .$$

We are mainly interested in the flows that are bounded below by a plane, say $y = 0$, and above by a free surface $y = h(x)$; and we wish to find the permissible h's consistent with the governing equations.

Two approaches have been used for the investigation of this problem. They are mainly concerned with the inviscid case and they lead to exactly the same result. The effects of the viscosity are only explored in a qualitative manner.

One approach is that of Benjamin and Lighthill[3]. Based on the assumption of irrotationality of the flow, they take

$$\psi = yf(x) - \frac{y^3}{3!} f''(x) + \frac{y^5}{5!} f'''(x) \ . \ . \ . \quad . \tag{8}$$

Retaining two terms in this series, they argue that the quantity

$$S = \int_0^h \left(\frac{p}{\rho} + \psi_y^2 \right) dy = \int_0^h \left[R - gy - \frac{1}{2}\psi_x^2 + \frac{1}{2}\psi_y^2 \right] dy$$

is a constant, where

$$R = \frac{p}{\rho} + \frac{1}{2}(\psi_x^2 + \psi_y^2) + gy$$

is also a constant. The resulting equation has the form

$$\frac{1}{3} Q^2 \left(\frac{dh}{dx} \right)^2 + gh^3 - 2Rh^2 + 2Sh - Q^2 = 0 \quad , \tag{9}$$

where

$$Q = uh$$

is also a constant.

Various features of cnoidal and solitary waves are contained in Eq. (9) and they have been fully discussed by Benjamin and Lighthill[3].

The other approach is that due to Serre[2]. He takes

$$\psi = QF(y/h(x)) \quad , \tag{10}$$

and then he computes p from the inviscid form of Eq. (7). Then Eq. (9) can be obtained by evaluating either

$$S = \int_0^h \left(\frac{p}{\rho} + \psi_y^2 \right) dy \quad ,$$

or

$$R = \frac{1}{h}\int_o^h \left[\frac{p}{\rho} + gy + \frac{1}{2}(\psi_x^2 + \psi_y^2)\right] dy \quad .$$

Since to a first approximation we have

$$f(x) = \frac{Q}{h(x)}$$

it is not surprising that we obtain the same result by these two somewhat different approaches.

Making use of the assumption (10), we now would like to make full use of Eqs. (6) and (7). There is no loss of generality if we take $F(1) = 1$ and $F(o) = 0$. Further, since

$$u = \frac{Q}{h} F'\left(\frac{y}{h}\right) \quad , \quad v = \frac{Q}{h^2} yh'F'\left(\frac{y}{h}\right) \quad ,$$

we have $F'(o) = 0$, by the boundary condition at $y = 0$. Now, if we integrate Eq. (6) with respect to y from 0 to h, after neglecting terms which are important only for the damping of shorter waves (cf. Appendix A), we obtain:

$$\frac{d}{dx}\left\{\frac{b}{2}\frac{Q^2}{h^2} + gh + aQ^2\left[\frac{h''}{h} - \frac{1}{2}\frac{h'^2}{h^2}\right]\right\} = -\nu Q\left\{F'(1)\frac{d}{dx}\left(\frac{h'}{h^2}\right) + \frac{F''(o)}{h^3}\right\} \quad . \tag{11}$$

where

$$a = \int_o^1 z^2 F'(z)^2 dz \quad , \quad \text{and} \quad b = \int_o^1 F'(z)^2 dz \quad .$$

We can recover Eq. (9), if we neglect the viscous terms on the right hand side of (11). If we neglect only the first term in the right hand side of (11), then we obtain

$$\frac{b}{2}\frac{Q^2}{h^2} + gh + \frac{a}{2}\, Q^2\, \frac{d}{dh}\left(\frac{h'^2}{h}\right) = R \quad , \tag{12}$$

and

$$\frac{dR}{dx} = -\nu QF''(o)/h^3 \quad . \tag{13}$$

The last set is also obtained by Serre[(2)].

It can be seen that in order to obtain physically much more satisfying results, we should retain the first term rather than the second on the right hand side of (11). In fact, retaining the second term may lead to unrealistic results within the present model. For instance, we would expect that the flow will eventually be a smooth flow of constant finite depth, while from (11), we can never achieve this. We could obtain the asymptotic flow of constant finite depth by giving the flow bed an inclination in the gravity field. Then the asymptotic flow is the one controlled by the Reynolds number rather than the subcritical flow which is of interest here. On the other hand, if we only retain the first term on the right hand side of (11), the transition from supercritical flow to subcritical flow may be explicitly demonstrated, and the subcritical flow of constant depth is the asymptotic flow on this spatial scale.

Under this approximation, we then obtain from (11):

$$\frac{b}{2}\frac{Q^2}{h^2} + gh + aQ^2\left[\frac{h''}{h} - \frac{1}{2}\frac{h'^2}{h^2}\right] + \nu QF'(1)\,\frac{h'}{h^2} = R \quad , \tag{14}$$

where R is a constant.

Denote $f_1 = \frac{h}{h_o}$ and $f_2 = h'$, $\hat{x} = \frac{x}{h_o}$, the Reynolds number

$R_o = \frac{aQ}{\nu F'(1)}$, $\alpha = \frac{gh_o^3}{aQ^2} = \frac{gh_o}{au_o^2}$, $\beta = \frac{Rh_o^2}{aQ^2}$, where h_o is some reference depth and u_o the corresponding average velocity; then we can rewrite Eq. (14) as

$$\frac{df_1}{d\hat{x}} = f_2 = P_1(f_1, f_2) \quad , \tag{15}$$

and

$$\frac{df_2}{d\hat{x}} = \frac{1}{2}\frac{f_2^2}{f_1} - \frac{1}{R_o}\frac{f_2}{f_1} - \left[\alpha f_1^2 - \beta f_1 + \frac{b}{2a}\frac{1}{f_1}\right] = P_2(f_1, f_2) \quad . \tag{16}$$

The critical points of this autonomous system[6] are given by $P_1 = 0$ and $P_2 = 0$, or $f_2 = 0$ and

$$G(f_1) = \alpha f_1^2 - \beta f_1 + \frac{b}{2a}\frac{1}{f_1} = 0 \quad .$$

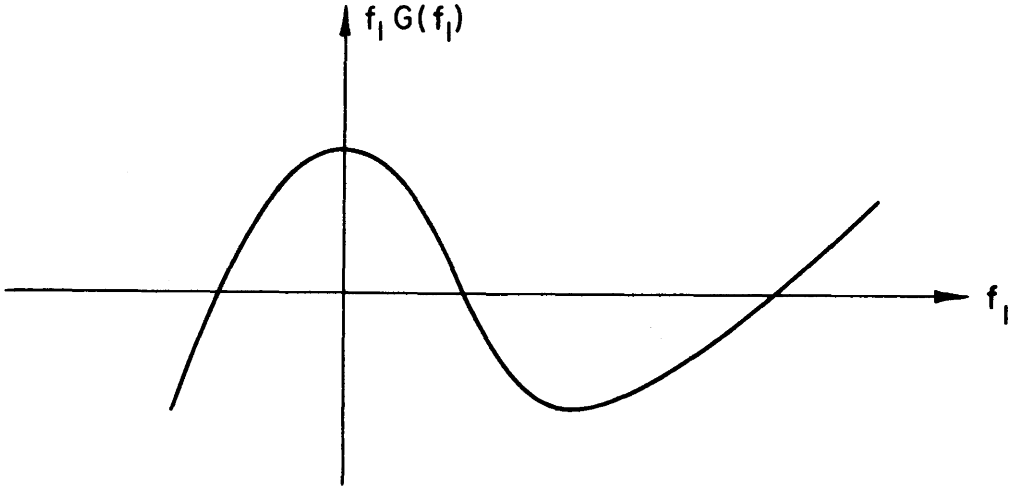

FIG. 2

The behavior of $f_1 G(f_1)$ is shown in Fig. 2. The minimum occurs at $f_1 = \frac{2\beta}{3\alpha}$, and $G(f_1)$ will have two positive zeroes if and only if $\frac{b}{2a} - \frac{4}{27}\left(\frac{\beta^3}{\alpha^2}\right) < 0$. This condition will always be fulfilled if at some initial section, where $h = h_o$, h' and h'' are all very small, then we have $\beta = \alpha + \frac{b}{2a}$ to a good approximation. With this approximation, the two zeros will degenerate to one when $\alpha = \frac{b}{a}$. These two zeroes we shall denote as $f_1^<$ and $f_1^>$ with $f_1^< < \frac{2\beta}{3\alpha}$ and $f_2^> > \frac{2\beta}{3\alpha}$.

Now the behavior of the solutions of Eqs. (15) and (16) about the critical points may be determined from the study of the characteristic equation[6]:

$$\lambda^2 - (A+D)\lambda + AD - BC = 0 \quad , \tag{17}$$

where

$$A = \frac{\partial P_1}{\partial f_1}, \quad B = \frac{\partial P_1}{\partial f_2}, \quad C = \frac{\partial P_2}{\partial f_1}, \quad D = \frac{\partial P_2}{\partial f_2} \quad .$$

In our case, at the critical points:

$$A = 0, \quad B = 1 \quad , \quad C = -2\alpha f_1 + \beta + \frac{b}{2a}\frac{1}{f_1^2} = -\alpha f_1 + \frac{b}{a}\frac{1}{f_1^2} \quad , \quad D = -\frac{1}{R_o f_1} \quad .$$

Thus the roots of Eq. (17) are given by

$$\lambda = \frac{1}{2}\left[-\frac{1}{R_o f_1} \pm \sqrt{\frac{1}{R_o^2 f_1^2} - 4\left(\alpha f_1 - \frac{b}{a}\frac{1}{f_1^2}\right)}\right] . \tag{18}$$

It is easy to see that

$$C = -\alpha f_1 + \frac{b}{a}\frac{1}{f_1^2} \begin{matrix} < 0 \\ > 0 \end{matrix} \quad \text{for} \quad \begin{matrix} f_1 = f_1^> \; , \\ f_1 = f_1^< \; . \end{matrix}$$

Therefore since the Reynolds number is usually quite large, we see that $f_1^<$ is a saddle point, while $f_1^>$ is a stable spiral point. The flow corresponding to $f_1^>$ is the subcritical flow which is stable; while the flow corresponding to $f_1^<$ is supercritical and unstable in it neighborhood.

Furthermore, by the Poincare-Bendixson theorem[6], we can conclude that $f_1^>$ is a limit point, to which the flow eventually approaches. However, when the viscous effect is small, the flow will experience "cnoidal wave" type of motion before finally approaching the smooth subcritical flow. Thus, although the transition from supercritical to subcritical flow is somewhat similar to the case of the shock, the detailed structure is much more complex.

When $h = h_o$, $h' = 0$, $h'' = 0$ at the initial section, then $\beta = \alpha + \frac{b}{2a}$. The two positive zeroes are then $f_1 = 1$, and $f_1 = \frac{1}{4\alpha}\left[\frac{b}{a} + \sqrt{\frac{b^2}{a} + \frac{8\alpha b}{a}}\right]$. Thus $f_1^< = 1$, and $f_1^> = \frac{1}{4\alpha}\left[\frac{b}{a} + \sqrt{\frac{b^2}{a^2} + \frac{8\alpha b}{a}}\right]$ if $\alpha < \frac{b}{a}$, and vice versa. Hence the critical flow is defined by $\alpha = \frac{b}{a}$, or

$$\frac{gh_o}{u_o^2} = b \quad . \tag{19}$$

For the parabolic velocity profile, we have

$$F'(z) = \frac{3}{2}\, z(2-z) \quad .$$

Thus

$$b = \int_0^1 F'^2(z)dz = 1.2 \quad ,$$

and the critical average velocity is

$$u_o = \sqrt{\frac{gh_o}{1.2}} \quad ,$$

while the critical surface velocity u_1 is

$$u_1 = 1.36\sqrt{gh_o} \quad .$$

On the other hand, for the uniform velocity profile, the critical velocities are given by

$$u_o = u_1 = \sqrt{gh_o} \quad .$$

IV. Vortex Breakdown

Vortex breakdown refers to the abrupt and drastic change of structure which sometimes occurs in a swirling flow. According to Harvey(4) and Benjamin(5), the breakdown appears to be the bridging flow between two rotating flows in a manner analogous to a hydraulic jump. If indeed such is the case, it should be of interest to see how the scheme we have employed in dealing with the hydraulic jump may be used for this case.

We shall be interested in the steady axisymmetric flow of an incompressible fluid. Then in the cylindrical coordinates, we have the following governing equations:

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{\partial v_z}{\partial z} = 0 \quad , \tag{20}$$

$$v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 v_r - \frac{v_r}{r^2}\right) \quad , \tag{21}$$

$$v_r \frac{\partial v_\theta}{\partial_r} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r} = \nu\left(\nabla^2 v_\theta - \frac{v_\theta}{r^2}\right) \quad , \tag{22}$$

$$v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu(\nabla^2 v_z) \quad , \tag{23}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \quad .$$

On account of Eq. (20), we may introduce the Stokes stream function ψ, so that

$$v_z = \frac{1}{r}\frac{\partial\psi}{\partial r} \quad , \quad v_r = -\frac{1}{r}\frac{\partial\psi}{\partial z} \quad .$$

In analogy with the hydraulic jump, we define here also a surface $r = h(z)$ which will mark the boundary of the breakdown region so that for $r < h$, we have

$$\psi = QF\left(\frac{r}{h}\right) \quad , \tag{24}$$

i.e., the velocity in each section is similar with the scale of h. For $r > h$, we could require conditions like

$$\psi = QG\left(\frac{r-h}{a-h}\right) \quad ,$$

where a is radius of the container, but the outer flow region is not of interest to us in general.

There is no loss of generality if one take $F(1) = 1$ and $F(o) = 0$. Further, since

$$v_z = \frac{Q}{rh} F'\left(\frac{r}{h}\right) \quad , \quad v_r = \frac{Q}{h^2} h'F'\left(\frac{r}{h}\right) \quad ,$$

to insure that the stress is finite at $r = 0$, we have $F'(x)=O(x^2)$ as $x \rightarrow 0$. The constancy of circulation, which we take to be valid here, will require that

$$v_\theta = \frac{1}{h} f\left(\frac{r}{h}\right) \quad . \tag{25}$$

Then, if we multiply Eq. (23) by r and integrate with respect to r for 0 to h, we obtain the following equation after making some appropriate approximations (cf. Appendix B):

$$\frac{d}{dz}\left\{\frac{bQ^2}{2h^4} - \frac{f^2(1)}{4h^2} + \frac{aQ^2}{2}\left[\frac{h''}{h^3} - \frac{h'^2}{h^4}\right] + \nu QF'(1)\frac{h'}{h^3}\right\} = -\frac{\nu QF''(o)}{h^2} \quad , \tag{26}$$

where

$$a = \int_0^1 wF'^2(w)dw \quad , \qquad \text{and} \qquad b = \int_0^1 \frac{F'^2(w)}{w}\, dw \quad .$$

The similarity between the last equation and Eq. (11) for the hydraulic jump is quite evident. A difference to be noted is that, while the free surface $y = h(x)$ in the case of hydraulic jump is definite, here the surface $r = h(z)$ is not yet so definite. We shall soon discuss this point.

We shall neglect the term on the right hand side of Eq. (26) since it will play a role only in the far downstream flow where this model is most probably no longer valid.

If, further the other viscous term can also be neglected, then Eq. (26) becomes

$$\frac{2bQ^2}{h^4} - \frac{f^2(1)}{h^2} + \frac{aQ^2}{h}\frac{d}{dh}\left(\frac{h'^2}{h^2}\right) = R \quad , \tag{27}$$

which will lead to

$$a^2Q^2h'^2 = \frac{R}{2}h^4 + f^2(1)h^2 \ln h + Sh^2 + bQ^2 = M(h) \quad , \tag{28}$$

where R and S are integration constant.

The last equation is quite similar to Eq. (9). However, there is some difference. Consider some initial section where h' and h'' are small, then we have

$$R < 0 \qquad \text{if} \qquad \frac{f^2(1)}{h_o^2} > \frac{2bQ^2}{h_o^4} \quad ,$$

$$R > 0 \qquad \text{if} \qquad \frac{f^2(1)}{h_o^2} < \frac{2bQ^2}{h_o^4} \quad .$$

The behavior of the curve $M(h)$ for these two cases is schematically shown in Fig. 3. The admissible solutions are for those values of h for which $M(h) > 0$. Thus we see that only for $R < 0$ do we have the breakdown and the cnoidal type of oscillation of the vortex region.

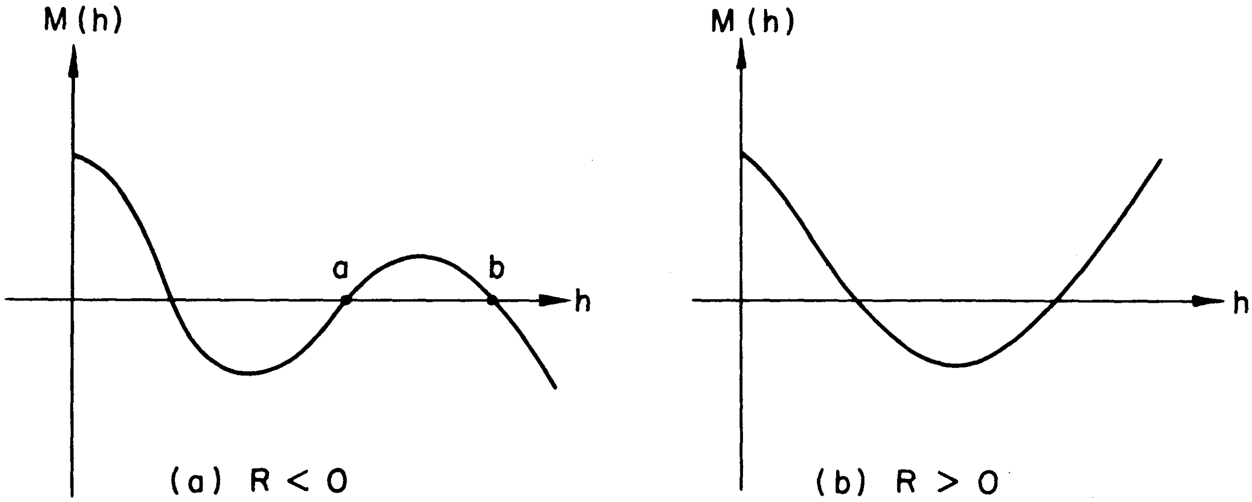

FIG. 3

The condition

$$\frac{f^2(1)}{h_o^2} > \frac{2bQ^2}{h_o^4} \quad , \tag{29}$$

is equivalent to $v_\theta^2(h_o) > v_z^2$ for the case of uniform initial flow when $F'(z) = 2z$.

To study the mechanism of the vortex breakdown, we go back to Eq. (26), again neglecting its right hand side. Then we have

$$\frac{2bQ^2}{h^4} - \frac{f^2(1)}{h^2} + 2aQ^2\left[\frac{h''}{h^3} - \frac{h'^2}{h^4}\right] + 4\nu QF'(1)\,\frac{h'}{h^3} = R \quad . \tag{30}$$

Let us write $f_1 = \frac{h}{h_o}$, $f_2 = h'$, $\hat{z} = \frac{z}{h_o}$, $R_o = \frac{aQ}{2\nu F'(1)h_o}$, $\alpha = \frac{f^2(1)h_o^2}{2bQ^2}$, $\beta = \frac{Rh_o^4}{2bQ^2}$, where h_o is some reference depth; then Eq. (30) can be rewritten as

$$\frac{df_1}{d\hat{z}} = f_2 = P_1(f_1, f_2) \quad , \tag{31}$$

$$\frac{df_2}{d\hat{z}} = \frac{f_2^2}{f_1} - \frac{b}{a}\left\{\frac{f_2}{R_o} - \beta f_1^3 - \alpha f_1 + \frac{1}{f_1}\right\} = P_2(f_1, f_2) \quad . \tag{32}$$

The critical points of this system are given by

$$f_2 = 0 \qquad \text{and} \qquad G(f_1) = \beta f_1^3 + \alpha f_1 - \frac{1}{f_1} = 0 \quad .$$

The zeros of $G(f_1) = 0$ are given by

$$f_1 = \pm\left\{\frac{1}{2\beta}\left[-\alpha \pm \sqrt{\alpha^2+4\beta} \;\right]\right\}^{\frac{1}{2}} \quad .$$

That there are sections for which h' and h'' are very small implies there are at least one real zero. Hence $\alpha^2 > -4\beta$. There are two positive zeros if $\beta < 0$ and only one positive zero if $\beta > 0$. We may take the initial section as the one with $h = h_o$ and with negligible h' and h''. Then $f_1 = 1$ is one zero of $G(f_1) = 0$. This is the only zero if $\beta = 1-\alpha > 0$. When $\beta = 1-\alpha < 0$, the other zero is $f_1^* = -\frac{1}{\beta} = \frac{1}{\alpha-1}$. It is clear that $f_1^* > 1$ if $\alpha < 2$ and vice versa. The behavior of $G(f)$ is schematically shown in Fig. 4.

Let us now consider the critical point

$$f_2 = 0 \qquad \text{and} \qquad f_1 = 1.$$

At this point

$$A = \frac{\partial P_1}{\partial f_1} = 0 \;, \qquad B = \frac{\partial P_1}{\partial f_2} = 1 \quad ,$$

$$C = \frac{\partial P_2}{\partial f_1} = \frac{b}{a}\{3\beta + \alpha + 1\} = \frac{2b}{a}\;\{2-\alpha\}$$

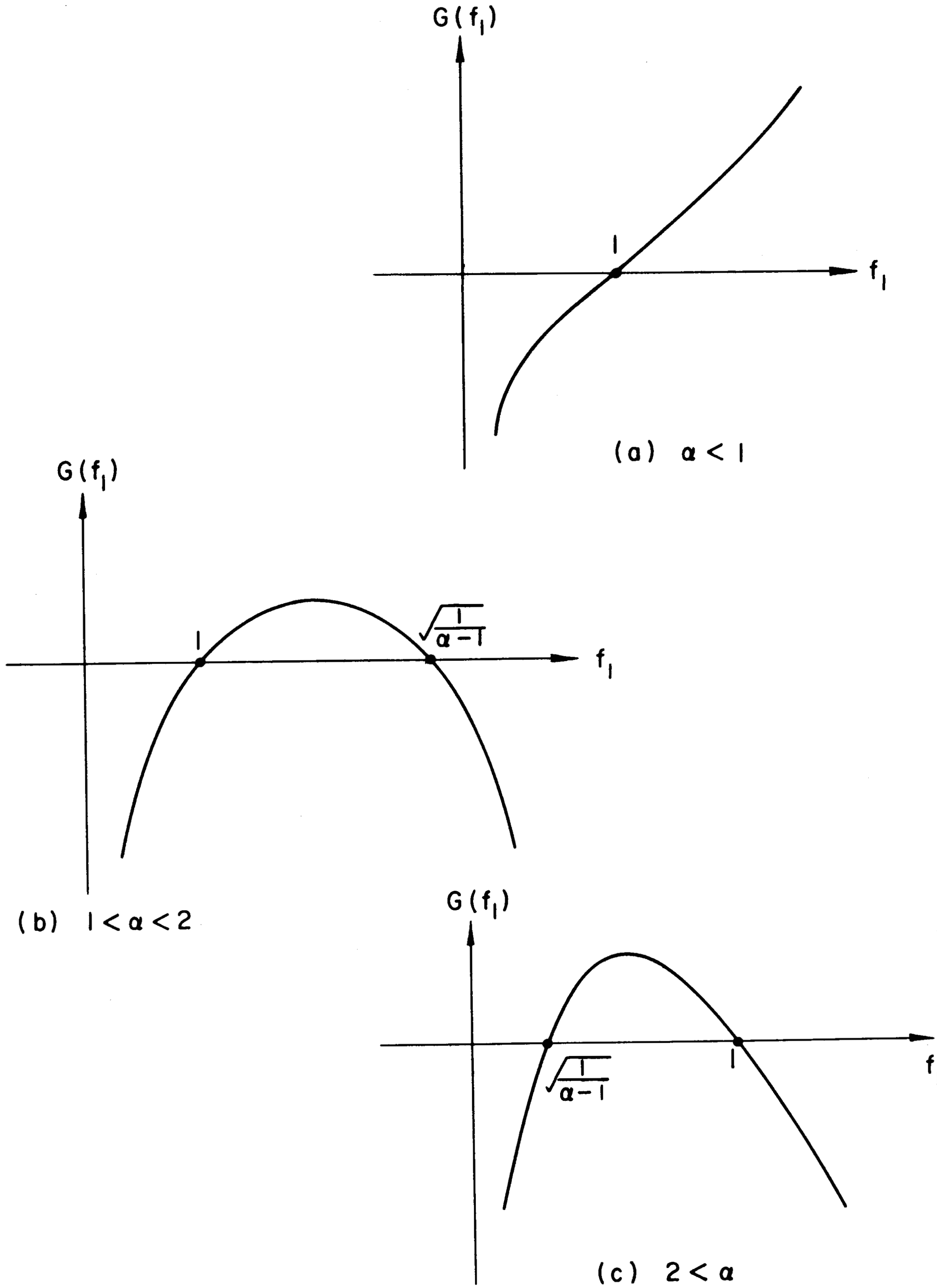
G(f_1)
1
f_1
(a) α < 1
G(f_1)
1
$\sqrt{\frac{1}{\alpha-1}}$
f_1
(b) 1 < α < 2
G(f_1)
$\sqrt{\frac{1}{\alpha-1}}$
1
f_1
(c) 2 < α

FIG. 4

$$D = \frac{\partial P_2}{\partial f_2} = - \frac{b}{aR_o} \quad .$$

Hence the roots of the characteristic Eq. (17) are

$$\lambda = \frac{1}{2}\left\{- \frac{b}{a}\frac{1}{R_o} \pm \sqrt{\frac{b^2}{a^2R_o^2} + \frac{8b}{a}(2-\alpha)}\right\} . \qquad (33)$$

When the Reynolds number is large, we conclude that this critical point is a stable spiral point if $\alpha > 2$, and it is a saddle point if $\alpha < 2$.

Thus when $\alpha > 1$, i.e. when condition is met for possible vortex breakdown, the critical flow is the one with $\alpha = 2$, or

$$\frac{f^2(1)h^2}{4bQ^2} = 1 \quad . \qquad (34)$$

For uniform axial flow, this means when

$$v_\theta^2(h) = 2v_z^2 \quad . \qquad (35)$$

The flow is supercritical if $\alpha < 2$, or $\frac{f^2(1)h^2}{4bQ^2} < 1$; and the flow is sub-critical if $\alpha > 2$, i.e. $\frac{f^2(1)h^2}{4bQ^2} > 1$.

A strange result is that in this context, the flow is unstable even for $\alpha < 1$, i.e. when no breakdown is supposed to be expected. We interpret this instability as the sensitivity of the flow to the downstream conditions. On the other hand, when there is large swirl present, whether there is vortex breakdown or not, the flow is quite insensitive to the downstream condition. Rotation of the flow system has thus a stabilizing effect.

To make our results more definite, we shall make the plausible assumption that the controlling stream surface $r = h(z)$ is to be chosen as the least critical one. The criticality of the whole system will be that of the least critical stream surface. The choice of this controlling stream surface is indeed not so arbitrary if we consider the phenomenon of vortex breakdown as an even closer analogy to the hydraulic jump of a stratified fluid. Indeed, even for ordinary hydraulic jumps, the free surface is also the least critical stream surface. Then, for uniform axial flow, the controlling surface is that of maximum swirl velocity, and the critical swirl angle would be $\tan^{-1} \frac{1}{2} \approx 35°$, while no breakdown in the sense a supercritical flow will go over to a finite subcritical flow is expected for swirl angle larger than 45°.

V. Discussions

We may believe with confidence that our treatment of hydraulic jump reveals the essential features of the problem. The main reason for this view is that to the first approximation of long waves, our stream function includes the inviscid potential result. We can not claim confidence to the same extent for the problem of the vortex breakdown. For rotational axisymmetrical inviscid flows, the stream function satisfies the following equation[5][7]

$$D^2 \psi = r^2 \frac{dH}{d\psi} - \frac{KdK}{d\psi} , \tag{36}$$

where $D^2 = \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r}$, and $K(\psi) = rv_\theta$. Far before and after the breakdown, when the stream tubes are straight, ψ should satisfy

$$\left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r}\right)\psi = r^2 \frac{dH}{d\psi} - \frac{KdK}{d\psi} .$$

Benjamin's paper is mainly devoted to the study of this multitude of possible solutions. It is clear that the supercritical and subcritical flows would not in general obey the similarity rule (24) even in the first approximation due to the presence of vorticity. However, it is our opinion that our treatment still can be considered as a good approximation since we are only interested in the gross features of the phenomenon. Benjamin's approach is needed if we want to study the detailed flows. But then the mechanisms of breakdown would indeed by very difficult to treat. An improvement to our treatment could presumably be obtained if instead of (24) we take

$$\psi = QF\left(\frac{g(r)}{g(h)}\right) \quad ,$$

where $\psi = g(r)$ satisfies

$$r^2 \frac{dH}{d\psi} - \frac{KdK}{d\psi} = 0 \quad ,$$

i.e. $g(r)$ is the approximate solution of (36) for large swirl in the initial section. However, we do not expect that it will alter much the essential results.

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Appendix A

With $\psi = QF\left(\frac{y}{h(x)}\right)$, we have

$$u = \frac{Q}{h} F'\left(\frac{y}{h}\right), \qquad v = \frac{Qyh'}{h^2} F'\left(\frac{y}{h}\right) .$$

Then it may be readily found that

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = Q^2 \left[\frac{yh''F'^2}{h^3} - \frac{yh'^2F'^2}{h^4} \right] ,$$

and

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = Q \left[\frac{yh'''F'}{h^2} - \frac{6yh''h'F'}{h^3} - \frac{3y^2h''h'F''}{h^4} + \frac{6yh'^3F'}{h^4} + \frac{6y^2h'^3F''}{h^5} \right.$$

$$\left. + \frac{y^3h'^3F'''}{h^6} + \frac{2h'F''}{h^3} + \frac{yh'F'''}{h^4} \right] .$$

Thus, using condition $p = 0$ on $y = h(x)$, we obtain

$$\frac{p}{\rho} = gh - gy + Q^2\left(\frac{h''}{h} - \frac{h'^2}{h^2}\right)\int_{\frac{y}{h}}^{1} zF'^2(z)dz - \nu Q\left[\left(h''' - \frac{6h''h'}{h} + \frac{6h'^3}{h^2}\right)\int_{\frac{y}{h}}^{1} zF'(z)dz\right.$$

$$-\left(\frac{3h''h'}{h} - \frac{6h'^3}{h^2}\right)\int_{\frac{y}{h}}^{1} z^2F''(z)dz + \frac{h'^3}{h^2}\int_{\frac{y}{h}}^{1} z^3F'''(z)dz + \frac{h'}{h^2}\left(2\int_{\frac{y}{h}}^{1} F''(z)dz + \int_{\frac{y}{h}}^{1} zF'''(z)dz\right.$$

In the following, we shall neglect terms with combined degree and orders 3 or more, i.e. terms like h''', $h'h''$, and h'^3, in the expression for $\frac{p}{\rho}$. Thus we have essentially made a long wave approximation. Since these terms are all associated with ν, this approximation means that we have neglected some damping terms for the shorter waves, but not the generating terms for the shorter waves.

Also, we have

$$u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} = -\frac{Q^2}{h^3}\,h'F'^2 \quad ,$$

and

$$\nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \nu Q\left[\left(\frac{2h'^2}{h^3} - \frac{h''}{h^2}\right)F' + \left(\frac{4h'^2}{h^4} - \frac{h''}{h^3}\right)yF'' + \frac{h'^2}{h^5}\,y^2F''' + \frac{F'''}{h^3}\right] \quad .$$

It is readily verified that

$$\int_0^{h(x)} dy\,y^n\left\{\frac{\partial}{\partial x}\left[f(x)\int_{\frac{y}{h}}^{1} g(z)dz\right]\right\} = h^n\left[fh' + \frac{hf'}{n+1}\right]\int_0^1 y^{n+1}\,g(y)dy \ . \qquad \text{(A-1)}$$

Also

$$\int_0^1 y^2F'''(y)dy = F''(1) - 2\int_0^1 yF''(y)dy = F''(1) - 2[\,F'(1) - 1\,] \quad . \text{(A-2)}$$

We shall here make use of the approximation that $F''(1) = 0$, i.e. $\frac{\partial u}{\partial y} = 0$, on $y = h$. Therefore, if we integrate Eq. (6) with respect to y from 0 to h, we obtain

$$-\frac{bQ^2}{h^2}\,h' = -ghh' - aQ^2\left[h''' - \frac{2h''h'}{h} + \frac{h'^3}{h^2}\right] + \nu Q\left[F'(1)\left(\frac{2h'^2}{h^2} - \frac{h''}{h}\right) - F''(o)\,\frac{1}{h^2}\right],$$

or

$$\frac{d}{dx}\left\{\frac{b}{2}\,\frac{Q^2}{h^2} + gh + aQ^2\left[\frac{h''}{h} - \frac{1}{2}\,\frac{h'^2}{h^2}\right] + \nu QF'(1)\,\frac{h'}{h^2}\right\} = -\frac{\nu QF''(o)}{h^3} \quad , \qquad \text{(A-3)}$$

where

$$a = \int_0^1 z^2F'^2(z)dz \quad ,$$

and

$$b = \int_0^1 F'^2(z)dz \quad .$$

Appendix B

With

$$v_r = \frac{Q}{h^2} h'F'\left(\frac{r}{h}\right) , \qquad v_z = \frac{Q}{rh} F'\left(\frac{r}{h}\right) ,$$

it may be readily found that

$$v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} = \frac{Q^2}{rh^3} F'^2\left[h'' - \frac{2h'^2}{h}\right] ,$$

and

$$\nabla^2 v_r - \frac{v_r}{r^2} = Q\left\{\left[\frac{h'''}{h^2} - \frac{6h''h'}{h^3} + \frac{6h'^3}{h^4} - \frac{h'}{r^2h^2}\right]F' + \left[-\frac{3rh''h'}{h^4} + \frac{6rh'^3}{h^5} + \frac{h'}{rh^3}\right]F'' + \left[\frac{r^2h'^3}{h^6} + \frac{h'}{h^4}\right]F'''\right\} .$$

Thus, from Eq. (21):

$$\frac{p}{\rho} = \frac{p(h)}{\rho} - \frac{1}{h^2}\int_{r/h}^{1} \frac{f^2(w)}{w}\, dw + \frac{Q^2}{h^3}\left(h'' - \frac{2h'^2}{h}\right)\int_{r/h}^{1} \frac{F'^2(w)}{w}\, dw$$

$$- \nu Q\left\{\left[\frac{h'''}{h} - \frac{6h''h'}{h^2} + \frac{6h'^3}{h^3}\right]\int_{r/h}^{1} F'(w)dw\right.$$

$$- \frac{h'}{h^3}\int_{r/h}^{1} \frac{F'(w)}{w^2}\, dw + \left[-\frac{3h''h'}{h^2} + \frac{6h'^3}{h^3}\right]\int_{r/h}^{1} wF''(w)dw + \frac{h'}{h^3}\int_{r/h}^{1} \frac{F''(w)}{w}\, dw$$

$$\left. + \frac{h'^3}{h^3}\int_{r/h}^{1} w^2F'''(w)dw + \frac{h'}{h^3}\int_{r/h}^{1} F'''(w)\, dw\right\} . \qquad \text{(B-1)}$$

In the following, we shall neglect terms with combined degrees and orders 3 or more, as in Appendix A. Also, for p(h), we shall make use of the inviscid result:

$$H(\psi) = \frac{p}{\rho} + \frac{1}{2}[v_r^2 + v_\theta^2 + v_z^2] \quad . \tag{B-2}$$

Now the last equation is the inviscid result of the full set of equations (20) to (23). It imposes too much information within this scheme. From our experience with the problems of normal shock and hydraulic jump, we are inclined to consider that the effects of v_r on $\frac{p}{\rho}$ have already been included in (B-1), while the effects of v_z on $\frac{p}{\rho}$ will be the content of Eq. (23) which we shall soon deal with. Thus instead of (B-1), we shall rather take:

$$\hat{H}(\psi) = \frac{p}{\rho} + \frac{v_\theta^2}{2} \quad . \tag{B-3}$$

Then we obtain

$$\frac{p(h)}{\rho} = \hat{H}(Q) - \frac{1}{2}\frac{f^2(1)}{h^2} \quad .$$

That we apply (B-3) to the surface $r = h$, instead of other stream surfaces is due to the consideration that $r = h$ is a controlling stream surface, and our model is closest to reality near $r = h$. Thus using (A-1) and (A-2), we obtain

$$\int_0^h \frac{1}{\rho}\frac{\partial p}{\partial z}\, r dr = \frac{f^2(1)}{2}\frac{h'}{h} + \frac{Q^2}{2}\left[\frac{h'''}{h} - \frac{5h''h'}{h^2} + \frac{4h'^3}{h^3}\right]\int_0^1 wF'^2(w)dw$$

$$+ \frac{\nu Q}{2}\left(\frac{h''}{h} - \frac{h'^2}{h^2}\right)\left[F'(1) - F''(1)\right] \quad .$$

It is also readily obtained that

$$v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} = -\frac{2Q^2F'^2}{r^2h^3}\, h' \quad ,$$

and

$$\nabla^2 v_z = Q\left\{\left[-\frac{h''}{rh^2}+\frac{2h'^2}{rh^3}+\frac{1}{r^3h}\right]F' + \left[-\frac{h''}{h^3}+\frac{4h'^2}{h^4}-\frac{1}{r^2h^2}\right]F'' + \left[\frac{rh'^2}{h^5}+\frac{1}{rh^3}\right]F'''\right\}$$

Therefore, if we multiply Eq. (23) by r, then integrate with respect to r from 0 to h, we obtain, after using (A-2):

$$-2b\,\frac{Q^2h'}{h^3} = -\frac{f^2(1)}{2}\,\frac{h'}{h} - \frac{aQ^2}{2}\left[\frac{h'''}{h} - \frac{5h''h'}{h^2} + \frac{4h'^3}{h^3}\right]$$

$$-\frac{\nu Q}{2}\left[(3F'(1)-F''(1))\,\frac{h''}{h} - (5F'(1)+F''(1))\,\frac{h'^2}{h^2} + \quad\right], \quad \text{(B-4)}$$

where

$$a = \int_o^1 wF'^2(w)dw \quad , \qquad b = \int_o^1 \frac{F'^2(w)}{w}\, dw \quad ,$$

and

$$c = -\int_o^1 F'''(z)dz + \int_o^1 \frac{F''(z)}{z}\,dz - \int_o^1 \frac{F'(z)}{z^2}\,dz = F''(o)-F''(1)+F'(1) \quad ,$$

if the behavior of $F(z)$ near $z = 0$ is to make the integrals exist.

Now although $F(z)$ may take quite general form, we want to consider these cases in which v_z is uniform in the neighborhood of $r = h$; thus we make $F'(1) = F''(1)$. This approximation, as we may appreciate is really not a very restrictive one. Then (B-4) becomes

$$-\frac{2bQ^2h'}{h^3} = -\frac{f^2(1)}{2}\,\frac{h'}{h} - \frac{aQ^2}{2}\left[\frac{h'''}{h} - \frac{5h''h'}{h^2} + \frac{4h'^3}{h^3}\right]$$

$$-\nu Q\left[F'(1)\left(\frac{h''}{h} - \frac{3h'^2}{h^2}\right) + \frac{F''(o)}{h^2}\right] \quad ,$$

or

$$\frac{d}{dx}\left\{\frac{bQ^2}{2h^4} - \frac{f^2(1)}{4h^2} + \frac{aQ^2}{2}\left[\frac{h''}{h^3} - \frac{h'^2}{h^4}\right] + \nu QF'(1)\,\frac{h'}{h^3}\right\} = -\frac{\nu QF''(o)}{h^2}\,. \quad \text{(B-5)}$$

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The normal shock, hydraulic jump, and vortex breakdown have a common feature: they are all marked by a transition from a supercritical to a subcritical flow state. These phenomena are due to the nonlinearity of the flow, and it will be shown that viscosity also plays an essential role. This paper demonstrates explicitly how viscosity enters into these flows. The treatment of normal shock served a starting illustration. Then an equation governing the phenomenon of hydraulic jump is derived with the inclusion of the effect of viscosity. It is explicitly shown that supercritical flow is not stable and has to go through a transition to a conjugate subcritical flow state downstream. Similar treatment is also applied to vortex breakdown with largely similar results.

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